NUMERICAL ANALYSIS OF STRATEGIC CONTINGENT CLAIMS MODELS

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ABSTRACT

We study the numerical properties of a class of models recently introduced to calculate the values of corporate bonds and other corporate liabilities. Starting from a discrete-time extensive form game representing the consequences of financial distress, these "strategic contingent claims models" are associated with a particular free-boundary problem. Here we consider the properties of alternative solution techniques applied to this problem. We discuss four solution techniques of the finite difference type: explicit solutions, explicit solutions of the log transformed model, implicit solutions on a regular grid, and dynamically remeshed implicit solutions. To our knowledge this last method has not previously been employed in financial applications. We find that the use of dynamic remeshing can speed calculation solutions enormously. This opens the way to applying strategic contingent claims models in practical applications.

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1 Introduction

In this paper we explore the numerical properties of a class of models that have recently been introduced to value corporate bonds and other financial instruments in the presence of credit risk. This class of models is described as "strategic contingent claims analysis" in order to distinguish it from the line models for corporate bond valuation started by Robert Merton (1974) and subsequently developed by Black and Cox (1976), Jones, Mason, and Rosenfeld (1984) and others. The crucial question in valuation in the face of credit risk is how to model the lower reorganization boundary of the firm, that is, to identify those states of the world where bankruptcy occurs and the creditors take possession of the firm or otherwise receive through the courts a settlement of their claims. In "classical contingent claims", such as Merton's, this lower reorganization boundary is given exogenously. This line of research has continued with modifications in the definition of the boundary, which in some cases is treated as stochastic (e.g., Nielsen et al. (1993) or Longstaff and Schwartz (1995)); however, in all this work, the boundary is exogenously given and does not depend upon actions taken either by debtors or creditors. In strategic contingent claims analysis, these agents actions that are permitted under the bankruptcy regime in force are explicitly taken into account. These models are "strategic" in the sense that each agent is assumed to maximize payoffs given the strategies of the other players. Anderson and Sundaresan (1996) are the first to introduce a fully dynamic valuation model of this sort. Subsequently, Mella-Barral and Perraudin (1997), Mella-Barral (1995), and Anderson, Sundaresan, and Tychon (1996) have contributed to this type of modelling.

Strategic contingent claims analysis has a number of advantages over classical contingent claims in the modeling corporate bonds. In particular, it is possible to incorporate charateristics of actual bankruptcy laws and, in this way, to respect the stylized facts that have been established in the recent empirical literature on financial distress (e.g., Franks and Torous (1989, 1994) and Weiss (1990)). For example, the model of Anderson and Sundaresan (1996) is characterized by deviations from absolute priority of claims. Furthermore, if the bankruptcy process endows the debtor with relatively large amounts of bargaining power, the model generates realistic yield spreads over riskless claims without assuming unrealistic levels either of bankruptcy costs or of the volatility of the assets of the firm.

These advantages come at a cost. Anderson and Sundaresan solve their model numerically on a binomial lattice and find that obtaining accurate solutions for relatively simple security design problems can require substantial calculation time. In the face of this difficulty, and building on the contribution of Mella-Barral and Perraudin and on the real options literature more generally (see e.g., Dixit and Pindyck (1994)), Anderson, Sundaresan and Tychon (1996) show that it is possible to recast the Anderson-Sundaresan model in continuous time and thereby to recover some of the facility of classical contingent claims analysis. For example, it is possible to obtain analytical solutions under the assumption that the instrument is a perpetual coupon bond. Furthermore, for more realistic problems, the analyst can call upon diverse techniques for solving partial differential equations. However, strategic contingent claims models pose difficulties because the associated free boundary problem is potentially more difficult to solve than that of the exogenously-given boundary of classical strategic claims analysis.

In this paper we compare several algorithms for solving the Anderson-Sundaresan PDE. Explicit finite difference methods, working with either bond prices or logarithms of bond prices, are potentially unstable so that means for obtaining stable solutions to an acceptable degree of accuracy can greatly inflate computing time for most interesting problems. This leads us to consider implicit solutions. We first introduce an algorithm for solving our free boundary problem on a fixed regular grid. That requires less effort than explicit methods for most problems. However, to price long-dated bonds with acceptable accuracy can still require much computer time. Finally we introduce a new algorithm that calculates implicit finite difference solutions on a variable grid in the space dimension that is dynamically updated in the time dimension. For some interesting problems, this leads to speed gain of a factor of 200 over explicit methods.

The paper is organized as follows. Section 2 introduces the strategic contingent claims free boundary problem and establishes a useful characterization of the firm's lower reorganization boundary. Section 3 presents the algorithms that are used to solve this problem including the dynamically updated, variable grid method that we believe to be new in the area of computational finance. In section 4 we discuss the numerical results obtained with the alternative algorithms. Section 5 presents our conclusions on the importance of the results for further research in this area.

2 Strategic contingent claims valuation in discrete and continuous time

In this section we introduce strategic contingent claims models, briefly recalling the model of risky debt introduced in discrete time by Anderson and Sundaresan (1996). For models of this type with the time periods allowed to become arbitrarily short, we show how the valuation relations give rise to a partial differential equation (PDE) which is identical to the PDE which is found by Black and Scholes (1973) except that one term, the cash payout, and the boundary conditions must be found with reference to the posited bankruptcy game. In the case of the Anderson-Sundaresan model, the cash payout is determined by two simple equations, each applicable in distinct regions of the state space that are separated by a critical value V_t^*. This value can be interpreted as the "bankruptcy point" in that it is the highest value of firm assets leading the borrower to default on the terms of the financial contract. We then show that V_t^* is monotone decreasing in the time to the maturity of the contract and discuss the implications of this result.

Consider an owner/manager who uses debt to finance a project that generates a stream of uncertain rents as long as it is active. If the shareholder defaults on the debt contract, the bondholder may take the firm to bankruptcy court, which would involve a costly liquidation of the firm's assets to pay the outstanding principal. Since bankruptcy is optional, a strategic default is possible in which the shareholder pays an amount less than the contractual debt service but adequate to persuade the bondholder not to liquidate the firm. Thus, once the debt contract has been issued and the firm is in place, the relations of the shareholder and the bondholder are represented by an extensive-form game that can be described as follows.

The firm's assets-in-place follow a geometric Brownian motion process that, in discrete-time, is represented by V_{t-h} branching after a period of length h to either uV_{t-h} or dV_{t-h} with $d = 1/u$. After observing this state variable, the shareholder chooses a level of debt service S_t. If this is at least as great as the contractually specificed debt service, which for a straight bond is just the coupon rate c per year times the period length h times the principal P, the game advances to the next period. At the maturity date T, the contractual debt service is the principal plus interest, $(1 + ch)P$. If for $t < T$, the debtor default $(S_t < chP)$, a decision node for the bondholder is created. If he accepts, the game continues. If he rejects, the firm is liquidated and the bondholder is awarded the value of his collateral $Min(P, Max(V_t -$

$K, 0)$), where K is a constant liquidation cost.

This repeated bankrupcty game gives the scope of strategic default of the terms of the debt contract. The intuition behind this can be understood by considering the case of a contract with a principal of 100, a bankruptcy cost of 20, and an underlying asset value of 110 in the final period. Even though the debtor has adequate assets to fulfill the contract, he will default and pay only 90, which the bondholder accepts because this leaves him at least as well off as if he declares bankruptcy. In effect, the owner uses his bargaining power to extract a surplus. In earlier periods, the same logic applies. The complication is that, in deciding whether to accept or refuse a debt service less than the contractual amount, the bondholder must assess the expected value of continuing to hold the bond. If the underlying asset value process can be replicated period by period using existing assets, then both borrowers and creditors will assess continuation payoffs using the same martingale equivalent measure. Under this assumption, Anderson and Sundaresan solve for the subgame perfect equilibrium. The equilibrium level of debt service is,

$$S(V_t) = Min\left(chP, Max\left(0, Min(Max(V_t - K, 0), P) - \frac{\pi B_u + (1-\pi)B_d}{R}\right)\right), \tag{1}$$

where π is the martingale probability of an up-move, B_u and B_d are the values of the bond after an up-move or down-move, respectively, and R is the value of a dollar capitalized one period at the risk-free rate. The equilibrium value of debt thus is,

$$B = S(V_t) + \frac{\pi B_u + (1-\pi)B_d}{R}. \tag{2}$$

Obtaining the current values of claims involves finding the appropriate expression for π and imposing the facts that, at maturity, the continuation values of the bond will be zero and the contractual debt service is the principal P. Then the expressions can be solved recursively, as in a standard binomial pricing model. Accurate solutions require choosing h sufficiently small; that is, using a sufficiently large number of time steps T/h. In many circumstances, e.g., in representing solutions graphically or in calculating sensitivities, it is necessary to calculate a large number of bond values, which can cause total computation time to rise significantly. This is particularly true for security design applications, where one chooses contractual features (e.g., coupon, time to maturity, amoritzation schedules, call features, conversion features) to maximize equity values subject to a funding constraint that the bond has a sufficiently high value. For example, the computation of design experiment reported by Anderson and Sundaresan (1996) in Table 4, which is relatively simple when compared to the types of decisions faced by real world issuer, nevertheless required 72 hours on a RISC 6000 machine.

For this reason it is interesting to consider the continuous time equivalent of the discrete time game described above. The basic valuation relation (2) that the value, including current dividend, of the claim equals the equilibrium flow payment to the claim plus the continuation payoff. The same relation would be found in any strategic contingent claims model in which the contractual debt service depends upon the current value of the state variable and time but not upon the *paths* of the state variable or the realised debt service. Differences among these models are captured by the functional form of the equilibrium service flow $S(V_t)$ and boundary conditions. Otherwise the formulation remains the same. All such models can be considered in continuous time by writing the variables the variables R, $S(V_t)$, π, B_u, and B_d as explicit functions of the time-step step h. Using standard arguments familiar in contingent claim analysis (see, Cox and Rubinstein (1985)), Anderson, Sundaresan and Tychon (1996) show that, as $h \to 0$, one gets the partial differential equation,

$$\frac{1}{2}\sigma^2 V^2 \frac{\partial^2 B}{\partial V^2} + (r-\beta)V\frac{\partial B}{\partial V} - \frac{\partial B}{\partial \tau} - rB + S^*(V_t) = 0, \qquad (3)$$

where β is the rate of cash flow and τ is the time to maturity. With the exception of the last term on the left hand side, this is the standard differential equation found by Black and Scholes that emerges repeatedly in the analysis of claims contingent on a state variable that follows a geometric Brownian motion. The only modification here is the presence of the service flow S^*, which has been determined by the analysis of an extensive form game. The value function $B(.,.)$ can be solved from this equation by taking into account the appropriate boundary conditions, which again are found with reference to the posited game form.

For the Anderson-Sundaresan model applied to straight bonds, a coupon rate of c per year and a principal P reimbursed at the maturity date T, the service flow expression (1) takes simple forms within specific regions of the state space. If $c < r$, there are two regions in the state space divided at time t by the critical point V_t^*. [2] For $K < V < V_t^*$, there is strategic debt service

[2]The case of $r < c$ is similar except that there will be a second critical point V^{**}, such

in an amount that leaves the debtholder indifferent between accepting and declaring a bankruptcy that would give him his collateral value. It is shown that $V_t^* < P + K$, so that in this region $B = V - K$. As a consequence, $\frac{\partial B}{\partial V} = 1$, $\frac{\partial^2 B}{\partial V^2} = \frac{\partial B}{\partial \tau} = 0$. Inserting these into (3) yields $S^*(V) = \beta V - rK$. For $V > V_t^*$, the shareholders make a contractual debt service flow of cP. To find the function $B(V, T-t)$ for the bond value in this region, we must solve (3) with the appropriate boundary conditions. One useful condition is that, as $V \to \infty$, the bond value tends to the default risk-free value of the bond. A second is the "value matching" condition, $B(V_t^*, T-t) = V_t^* - K$. Were the critical value V_t^* known, this would determine the solution fully. Here, however, the critical value must be determined as part of the solution. As is now well-known in the real-option literature (see, Dixit and Pindyck (1994)), the function must satisfy the "smooth pasting condition" $\frac{\partial B(V_t^*, T-t)}{\partial V} = 1$. This determines both the function $B(.,.)$ and the critical value V_t^*.

Anderson, Sundaresan, and Tychon obtain analytical solutions for bond values and V_t^* in the special case of perpetual bonds (i.e., where $T \to \infty$). An example of this solution is given by the dashed line in figure 1. Note that, for the parameters chosen, the critical value is $V^* \approx 0.51$. For firm values less than this, the value of the bond is the liquidation value given by the straight line $V - K$. Above that, it is given by the concave curve approaching the value of $B = 0.5$ asymptotically. At the critical point, the debt service makes a jump from $\beta V^* - rK \approx 0.021$ to $cP = .05$. That is, in strategic defaults the offered debt service is a small fraction of the contractual debt service.

In the case of a bond with a finite maturity or more general bonds with scheduled amortization, sinking funds, and call features, an analytical solution is not generally possible over the entire range of the state space, and some numerical method must be used. For the case of a bond with five years to maturity ($T = 5$), Anderson, Sundaresan, and Tychon use explicit finite differences to obtain the solution depicted by the solid line in Figure 1. The computational effort involved is approximately the same as solving the binomial model for a large number of initial values V_0.

Notice that for the five-year bond $V_t^* \approx 0.65$. Thus, it appears that, all else equal, the critical value of the firm for which the owner switches to strategic debt service is higher for the finite maturity bond than for the perpetual. In fact this is a general property. We have

that for $V > V^{**}$, there will also be a strategic default on the contract. See Anderson, Sundaresan and Tychon (1996) for a discussion.

Proposition 1 *Within the Anderson-Sundaresan model for straight debt, the bankruptcy point V_t^* is a decreasing function of the time to maturity.*

To prove this, suppose for some t that $V_{t-1}^* = V_t^*$. Then, by value matching, $B(V_{t-1}^*) = V_{t-1}^* - K$. Furthermore, strategic debt service is given by $\beta V_{t-1}^* - rK$. Using this in (2) gives,

$$B(V_{t-1}^*) = \beta V_{t-1}^* - rK + \frac{\pi B(uV_{t-1}^*) + (1-\pi)B(dV_{t-1}^*)}{1+r}.$$

Now a downmove next period leads to strategic debt service so that $B(dV_{t-1}^*) = dV_{t-1}^* - K$, whereas an upmove leads to contractual debt service so that $B(uV_{t-1}^*) < uV_{t-1}^* - K$, where the latter inequality follows from the concavity of bond price. It then follows that

$$B(V_{t-1}^*) < \beta V_{t-1}^* - rK + \frac{\pi(uV_{t-1}^* - K) + (1-\pi)(dV_{t-1}^* - K)}{1+r}.$$

Now, by the properties of the martingale equivalent measure, we have

$$V_{t-1}^* = \beta V_{t-1}^* + \frac{\pi uV_{t-1}^* + (1-\pi)dV_{t-1}^*}{1+r},$$

(See Anderson and Sundaresan (1996) for a proof). Furthermore, $rK + \frac{K}{1+r} > K$, so

$$B(V_{t-1}^*) < V_{t-1}^* - K,$$

which is a contradiction. Assuming $V_{t-1}^* > V_t^*$ leads to a contradiction by similar reasoning. This completes the proof.

Intuitively, the result says that at some levels of asset value of the firm, bondholders would tolerate cheating on debt service if the bond is short term but not if it is long term. The reason is that these bonds are at a discount to their principal value ($B < P$) and the longer term bond tends to stand at a greater discount than does the short term bond. If the discount is so large that the longer term bond is less than the collateral value $V - K$ and is not being fully serviced, bondholder would liquidate the firm. To avoid this the shareholder with the longer term bond outstanding must service the bond fully.

The fact that the bankruptcy point for finite maturities can differ substantially from that of perpetuals shows that studying finite maturity bonds by considering analytical solutions of perpetuals is no substitute for more accurate numerical solutions. The fact that the bankruptcy point is a monotone decreasing function of the time to maturity is useful for finding efficient numerical solutions. We exploit this in the algorithms introduced in the next section.

The monotonicity of V_t^* has economically important implications in a slightly modified version of the bankruptcy game. In the above model it is possible that in the region of contractual debt service (i.e., $V_t^* < V$), the contracted interest payment may exceed the cash flow (i.e., $cP > \beta V$), so that shareholders receive a negative dividend. Since financing debt service through issueing new equity is rare, Anderson and Sundaresan study a version of the model where all debt service is assumed to be paid from available cash flow. If the latter is insufficient to pay bondholders a satisfactory amount, the firm is liquidated. The continuous time version of this modified game can be represented by two critical values for the assets of the firm. One, V_t^*, is defined as above. The second, $V^+ = cP/\beta$, is the minimal asset value such that cash flow is sufficient to pay contractual debt service. Now there may be either two or three regions of state space. If $V_t^* > V^+$, there are two regions as above. If $V_t^* < V^+$, there are three regions. For $V < V_t^*$, there is strategic default; for $V_t^* < V < V^+$, there is forced liquidation; and for $V^+ < V$, there is contractual debt service. Note that V^+ is independent of time, whereas V_t^* is increasing in calendar time. Since it is reasonable to assume all debt contracts are respected immediately after issue, we note that any firm that engages in strategic default at some date will not incur a forced liquidation at a later date.

This completes our general discussion of the free boundary problem encountered in strategic contingent claims analysis. We now turn to efficient numerical methods for solving this problem.

3 Methods of solution

We consider only the finite difference methods. These all involve replacing the partial derivatives in (3) by finite difference equations.[3] To understand the differences between the various methods, it is useful to introduce some notation. Variables are observed on a two-dimensional grid. Let the time dimension be indexed by $i = 0, ..., N$ and the state space dimension by $j = 0, ..., M$ so that, for instance, $V_{i,j}$ is value of the assets of the firm in

[3] For a general introduction to the use of finite difference methods for classical contingent claims analysis see Hull (1993).

state j on date i. The *forward difference operator* in the space dimension is given by

$$\Delta_j x_{i,j} = x_{i,j+1} - x_{i,j};$$

the *backward difference operator* is given by

$$\nabla_j x_{i,j} = x_{i,j} - x_{i,j-1};$$

and the *centered difference operator* is

$$\delta_j x_{i,j} = \frac{x_{i,j+1} - x_{i,j-1}}{2}.$$

The finite difference methods most familiar in finance have fixed step sizes in both the time and space dimensions, so that $\Delta_j V_{i,j} = \nabla_j V_{i,j} = \delta_j V_{i,j}$, which we denote as Δv. We also have occasion to use variable mesh schemes, where the space steps have variable length. For all methods we consider, the time steps are regular, $\Delta t = T/N$.

3.1 Explicit finite difference solution

The most easily computed solution for the PDE for $V > V^*$ is obtained by the explicit finite difference method. This involves approximating (3) by

$$\frac{1}{2}\sigma^2 V_{i,j}^2 \frac{\delta_j(\delta_j B_{i+1,j}/\delta_j V_{i,j})}{\delta_j V_{i,j}} + (r-\beta)V_{i,j}\frac{\delta_j B_{i+1,j}}{\delta_j V_{i,j}} + \frac{\Delta_i B_{i,j}}{\Delta t} - rB_{i,j} + cP = 0. \quad (4)$$

After collecting terms this can be written as

$$B_{i,j} = a_j B_{i+1,j-1} + b_j B_{i+1,j} + c_j B_{i+1,j+1} + d, \quad (5)$$

where a_j,b_j, and c_j are constants that depend upon j but, if a constant grid is used, not i. This system is completed by the terminal condition giving the bond value at maturity T,

$$B_{N,j} = Min(Max(V_{N,j} - K, 0), P), \quad (6)$$

and the upper boundary condition that, for the maximum asset values considered, the bond is evaluated as a risk-free bond

$$B_{i,M} = (1 - e^{-r(T-i\Delta t)})\frac{cP}{r} + Pe^{-r(T-i\Delta t)}. \quad (7)$$

For $i < N$, (5) is applied for $j = M-1, ..., j_i^*$, where V_{i,j_i^*} is the numerical approximation of the bankruptcy point V_t^*. For $j < j_i^*$, we use

$$B_{i,j} = Max(V_{i,j} - K, 0). \tag{8}$$

The only complication is determining j_i^*, which is done as follows. At maturity the bankruptcy point is $V_T^* = P + K - \Delta v$. Thus, we set $j_N^* = V_T^*/\Delta v$. At earlier time steps, we initially set $j_i^* = j_{i+1}^*$ and compute $B_{i,j}$ for $j = M-1, ..., j_i^*$ using (5). We then check to verify that this satisfies numerically smooth pasting $|B_{i,j_i^*+1} - B_{i,j_i^*} - \Delta v| < \epsilon$, where ϵ is a small constant. If not, we appeal to the propostion above and set $j_i^* \leftarrow j_i^* - 1$, repeating the process until smooth pasting is satisfied.

At each time step, this method is rapid since (5) gives the solution explicitly without requiring a solution of a simultaneous system. The disadvantage of this method is that, if the time step size is not chosen carefully, results can oscillate and diverge from the true solutions. Stability requires a_j, b_j, and $c_j > 0$. In particular, this implies, $b_M = \frac{1-\sigma^2 M^2 \Delta t}{1+r\Delta t} > 0$ or

$$\Delta t < \frac{1}{M^2\sigma^2}. \tag{9}$$

This means that the number of time steps $N = T/\Delta t$ must increase with both the square of the fineness and the volatility of the asset process. For long-dated bonds with realistic levels of asset volatility, this can pose a very serious constraint. The other problem of a very small Δt is that it tends to exaggerate rounding error and should be avoided if possible.

A more convenient way for computing explicit solutions uses logarithmic transformations as follows. Let $Y = lnV$ and $W(Y,t) = B(V,t)$. Thus,

$$B_v = W_y e^{-y}$$

$$B_{vv} = (W_{yy} - W_y)e^{-2y}$$

$$B_t = W_t,$$

and the PDE becomes

$$\frac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial Y^2} + (r - \beta - \frac{1}{2}\sigma^2)\frac{\partial W}{\partial Y} + \frac{\partial W}{\partial t} - rW + cP = 0, \tag{10}$$

The finite difference equivalent of (10) is

$$\frac{1}{2}\sigma^2 \frac{\delta_j(\delta_j W_{i+1,j}/\delta_j Y_{i,j})}{\delta_j Y_j} + (r - \beta - \frac{1}{2}\sigma^2)\frac{\delta_j W_{i+1,j}}{\delta_j Y_j} + \frac{\Delta_i W_{i,j}}{\Delta t} - rW_{i,j} + cP = 0. \tag{11}$$

After collecting terms this becomes

$$B_{i,j} = aB_{i+1,j-1} + bB_{i+1,j} + cB_{i+1,j+1} + d. \tag{12}$$

The terminal and boundary conditions, as well as the algorithm, are similar to the explicit method. This method is even easier to compute than the explicit method in levels of firm value since the coeffients in (12) are independent of both i and j. Furthermore, problems of instability are reduced somewhat. The log explicit method is stable if a, b and $c > 0$. In particular this requires

$$\Delta t < \frac{\Delta y^2}{\sigma^2}. \tag{13}$$

This can be less restrictive than condition (9). For example, for a 30 year bond issued by a firm with asset volatility $\sigma^2 = .1$ and a fixed space step size equal to 1 percent in the neighborhood of $V = 1$, the explict method requires more than 1 million time steps, whereas the log explicit requires 30,000. Thus, in relative terms, the log explicit method is quite efficient. However, in absolute terms, it is still too time consuming to be of use for many interesting problems. This leads us to consider implicit methods.

3.2 Implicit finite difference solution with regular space grid

We now consider implicit solutions to the strategic contingent claims model. This involves making the following discrete substitutions in the basic PDE (3)

$$\frac{1}{2}\sigma^2 V_{i,j}^2 \frac{\delta_j(\delta_j B_{i,j}/\delta_j V_{i,j})}{\delta_j V_{i,j}} + (r-\beta)V_{i,j}\frac{\delta_j B_{i,j}}{\delta_j V_{i,j}} + \frac{\Delta_i B_{i,j}}{\Delta_i t_i} - rB_{i,j} + cP = 0. \tag{14}$$

After substituting for these finite difference operators in the case of a regular space grid we obtain,

$$a'_j B_{i,j-1} + b'_j B_{i,j} + c'_j B_{i,j+1} = B_{i+1,j} + cP\Delta t, \tag{15}$$

where

$$a'_j = \frac{\Delta t j}{2}(-\sigma^2 j + r - \beta),$$
$$b'_j = \Delta t \sigma^2 j^2 + r\Delta t + 1,$$
$$c'_j = \frac{\Delta t j}{2}(-\sigma^2 j - r + \beta).$$

The terminal and upper boundary conditions are as in the explicit method. The lower boundary condition comes from value matching

$$B_{i,j_i^*-1} = V_{i,j_i^*-1} - K.$$

The algorithm for this method is as follows: We set V^* and j_i^* initially as in the explicit method. Now at each time step, we must solve a linear system $Ax = y$, where A is an $(M - j_i^*) \times (M - j_i^*)$ matrix. If the smooth pasting condition is satisfied $|x_{j_i^*+1} - x_{j_i^*} - \Delta v| < \epsilon$, we set $B_{i,j} = x_j$ for $j = j_i^*, ..., M-1$ and $B_{i,j} = Max(V_j - K, 0)$ for $j = 1, ..., j_i^* - 1$, and we move to the next earlier time period $i-1$. If smooth pasting is not satisfied, we reduce j_i^* by one. We recompute A, y, and solve the linear system again. We repeat this until smooth pasting is satisfied. Note that at each time step, for a given M, this method is slower than the explicit method since there is more work involved in solving a linear system than in caculating explict solutions. However, it is unconditionally stable. Thus we can choose a much bigger time step size and reduce the number of time steps so that the total work may be reduced.

3.3 Implicit finite difference solution with variable, dynamic remeshing

To price accurately by the implicit method with a regular grid we must set M large, and this leads to expensive calculations. Ideally we would like to keep both M and N reasonably small without sacrificing accuracy, at least in the ranges of the state space that interests us most. This can be done by using a variable grid size in the space dimension and by allocating the majority of space nodes where they are needed most.

In general, finite differences with a variable space grid means $\Delta_j V_{i,j} \neq \bigtriangledown_j V_{i,j} \neq \delta_j V_{i,j}$. As a result, with a variable grid, (14) does not simplify to the same extent as in the case of a regular grid. With a variable grid we obtain

$$a_j'' B_{i,j-1} + b_j'' B_{i,j} + c_j'' B_{i,j+1} = B_{i+1,j} + cP\Delta t, \tag{16}$$

where

$$a_j'' = -\frac{\Delta t \sigma^2 V_{i,j}^2}{2\delta_j V_{i,j} \bigtriangledown_j V_{i,j}} + \frac{\Delta t (r-\beta) V_{i,j}}{\Delta_j V_{i,j} + \bigtriangledown_j V_{i,j}},$$

$$b_j'' = \frac{\Delta t \sigma^2 V_j^2}{2\delta_j V_{i,j}}\left(\frac{1}{\bigtriangledown_j V_{i,j}} + \frac{1}{\delta_j V_{i,j}}\right) + r\Delta t + 1,$$

$$c''_j = -\frac{\Delta t\sigma^2 V_{i,j}^2}{2\delta_j V_{i,j}\Delta_j V_{i,j}} - \frac{\Delta t(r-\beta)V_{i,j}}{\Delta_j V_{i,j} + \bigtriangledown_j V_{i,j}}.$$

Note that the system is in the same form as with constant grid size. The only difference is that the coefficients a'', b'', and c'' are somewhat more complicated than a', b' and c' and that they require the storage of the space mesh $V_{i,j}$ for the dates i and $i+1$.

Computationally, the most interesting segment of state space of the strategic debt service model is in the neighborhood of V_t^*. Here we wish to calculate derivatives of our function $B(V,t)$ accurately since V_t^* must be calculated using smooth pasting. Errors in identifying V_t^* can easily be perpetuated along the whole $B(V,t)$ curve. However, V_t^* varies with time (see proposition 1) so that the variable mesh appropriate near maturity ($i = N$) would not be appropriate for earlier time periods. In light of these considerations, we introduce a dynamic remeshing scheme. [4]

The basic idea of the algorithm can be understood with reference to Figure 2. For the maturity period N, points $V_{N,1}, V_{N,2}, ...V_{N,5}$ constitute a section of coarse grid, $V_{N,6}, ..., V_{N,10}$ a section of fine grid, and $V_{N,11}...V_{N,M}$ a section of coarse grid. These are chosen to have the critical point V_{N,j_N^*} close to the center of the fine grid. In prior periods, $i = N-1, N-2...$, the critical values V_t^* may be significantly lower, so that it may be necessary to shift the fine mesh down as indicated in Figure 2. Note that the coarse space steps size has been chosen to be an interger multiple of the fine space step size so that coarse nodes of the i mesh are always aligned with nodes (either coarse or fine) of the $i+1$ mesh.

Dynamic remeshing is linked to changes in V_t^* according to the following scheme. We set the initial ($i = N$) grid V with finest mesh centered at $V_N^* = P + K$. For each time step i, we first calculate $\Delta V_j, \delta V_j$, and $\bigtriangledown V_j$. Then we use these along with previous solutions $B_{i+1,j}$ and boundary conditions to calculate the coefficient matrix A'' and y in the linear system $A''x = y$ determined by (16). After solving this we check that the solution satisfies smooth pasting. If not, we reduce j_i^*, reset A'' and y, and solve the linear system again until smooth pasting is satisfied. If so, we set $B_{i,j} = x_j$ for $j = j_i^*, ..., M-1$ and $B_{i,j} = Max(V_j - K, 0)$ for $j = 1, ..., j_i^* - 1$. After this, we check to see if j_i^* has changed so significantly that V_{i,j_i^*} is far from the

[4]Dynamic remeshing is a subject of active current research in numerical analysis. For an introduction and examples of how dynamic remeshing can dramatically improve the accuracy of solutions, see Celia and Gray (1992).

center of the fine mesh. If yes, we remesh V and reset y and j_i^* accordingly. This mesh will then be used in the next iteration $(i-1)$. Note that some values of y in the new grid must be obtained by interpolation of the y's in the old grid.

The advantage of using variable grid size with dynamic remeshing is that we achieve the desired accuracy in a specific region of state space while keeping the total number of mesh points low. This reduces computational time. The disadvantage is the added complexity of the programming involved. However, in an application that calls for repeated calculation of a large number of bond values, this effort can easily be justified if the speed gain is large. In order to assess this gain, we carry out some numerical experiments in the next section.

4 Results

Here we use the methods just described to price straight bonds under various assumptions about underlying parameters such as the time to maturity T, bankruptcy costs K, and the volatility of the firm σ. We are interested in finding the relative computer time needed to obtain results of a given accuracy. In the context of strategic contingent claims analysis we are interested in the accuracy of both calculated bond prices $B(.,.)$ and the bankruptcy point V_t^*.

As discussed in section 3.1, the number of time steps involved calculating explicit solutions can quickly become very large as the number of points on the space mesh grows or as the volatility of the firm increases. Some initial calculations working with a coarse space mesh for a five year bond suggests that the time needed by the explicit method is 25 times that of the implicit method with a regular grid. The experiments we report here work with a much finer space mesh; however, this is the minimal fineness required to obtain solutions sufficiently accurate to be useful in practice.

The implicit methods (both regular and dynamically remeshed) are implemented with a constant time step of $\Delta t = 0.01$ throughout. With dynamic remeshing we experiment with different space steps for the coarse and fine meshes. We find that, to achieve continuity, the step size of the fine mesh should not be dramatically smaller than the coarse mesh. The results we report here are based on three levels of fineness with 4 fine steps per coarse step and 2 medium steps per coarse step.

Table 1 reports results of experiments involving various principals P and bankruptcy costs K. [5] The maturity is set at 1 year and the volatility is set at a relatively low level.[6] The results in Table 1 show that the results obtained by the four methods agreed quite well with each other. In all cases, the dynamically remeshed variable grid implicit method is fastest by a wide margin. The second fastest is the log explicit method, which requires approximately twice the computational effort as the dynamically remeshed implicit. The third is the implicit method on a regular grid (about four times slower than the fastest method. Finally, the regular explicit method is considerably slower than all other methods and approximately 65 times slower than the fastest one. The computational times are relatively insensitive to changes in P or K.

Table 2 reports experiments on σ^2 and T. The results reveal what we would expect from our discussions in section 3, namely that firm volatility and the time to maturity are very important determinants of computational time for explicit methods. Indeed, referring to stability condition (9) and to Table 1 we can see that with with $\sigma^2 = 0.1$ the work involved in the explicit method is approximately 200 times that of the dynamically remeshed implicit method, implying that it is practically infeasible to calculate the untransformed explicit solutions for a large T. Consequently, we have not reported any results for that method. For the logarithmic explicit method, passing from $\sigma^2 = 0.03$ to $\sigma^2 = 0.1$ increases the computational time by a factor of 3, independently of the value of T. As in the previous experiment, the dynamically remeshed explicit method is in all cases the fastest method. For lower values of σ^2 the log explicit is faster than the implicit on a regular grid. However, when $\sigma^2 = 0.1$ the reverse is true. Computer times for both implicit methods are relatively insensitive to variations of σ^2 and increase approximately proportionally with T.

5 Conclusion

In this paper we have studied the strategic contingent claims model from the perspective of numerical analysis. We have found that restating the model in continuous time gives rise to a free boundary problem whose careful study ultimately leads to solution techniques that can significantly speed the calculations. This in turn opens the way for applying this type of model to a

[5] All calculations were implemented in Matlab Version 3.5 on a Pentium computer.

[6] Anderson and Sundaresan (1996) find that $\sigma^2 = 0.1$ corresponds roughly to the kinds of volatility of industrial firms rated AA. Thus $\sigma^2 = 0.03$ might correspond to a utility.

variety of applications of considerable practical importance. For example, investment bankers designing securities would potentially like to explore the implications for specific choices of contractual features through security design experiments of the type reported by Anderson and Sundaresan (1996). Dynamically remeshed implicit finite difference solutions may well allow such experiments to be completed in less than an hour rather than the three days required by lattice techniques.

More generally our result illustrate how dynamically remeshed, variable grids can be introduced into finite difference problems arising in finance. This technique could prove helpful in other applications in finance and economics. Finally, our algorithms have emerged directly from the analysis of our model. This illustrates how understanding the theoretical properties of the model can lead to improved numerical efficiency in solution techniques.

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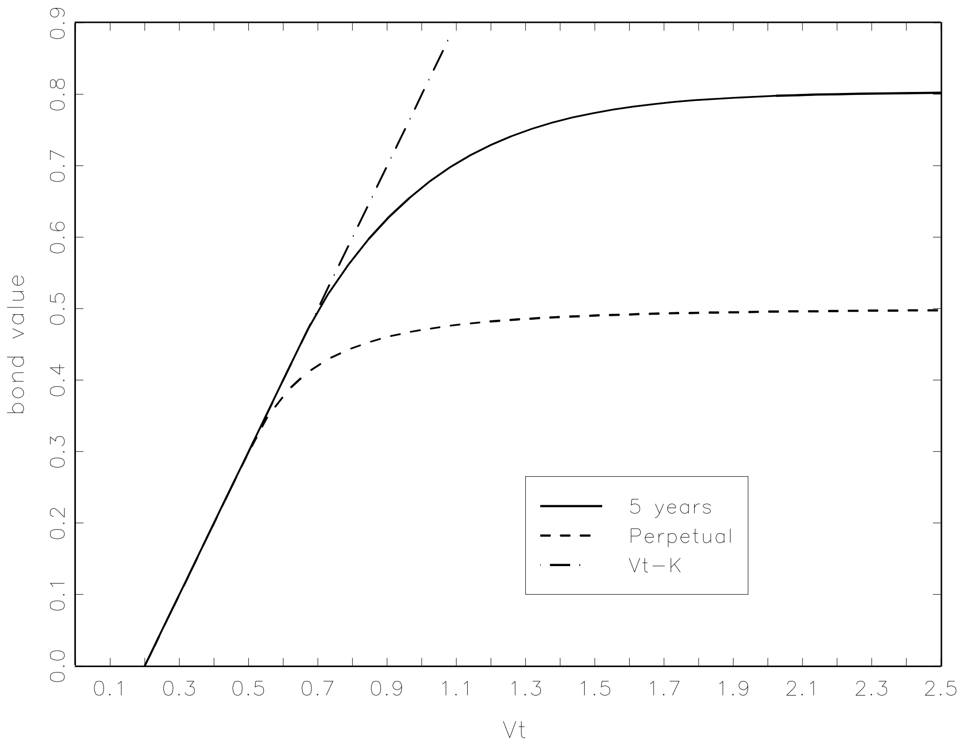

Figure 1 : $P = 1$, $c = 0.05$, $r = 0.1$, $\beta = 0.08$, $K = 0.2$, $\sigma^2 = 0.03$

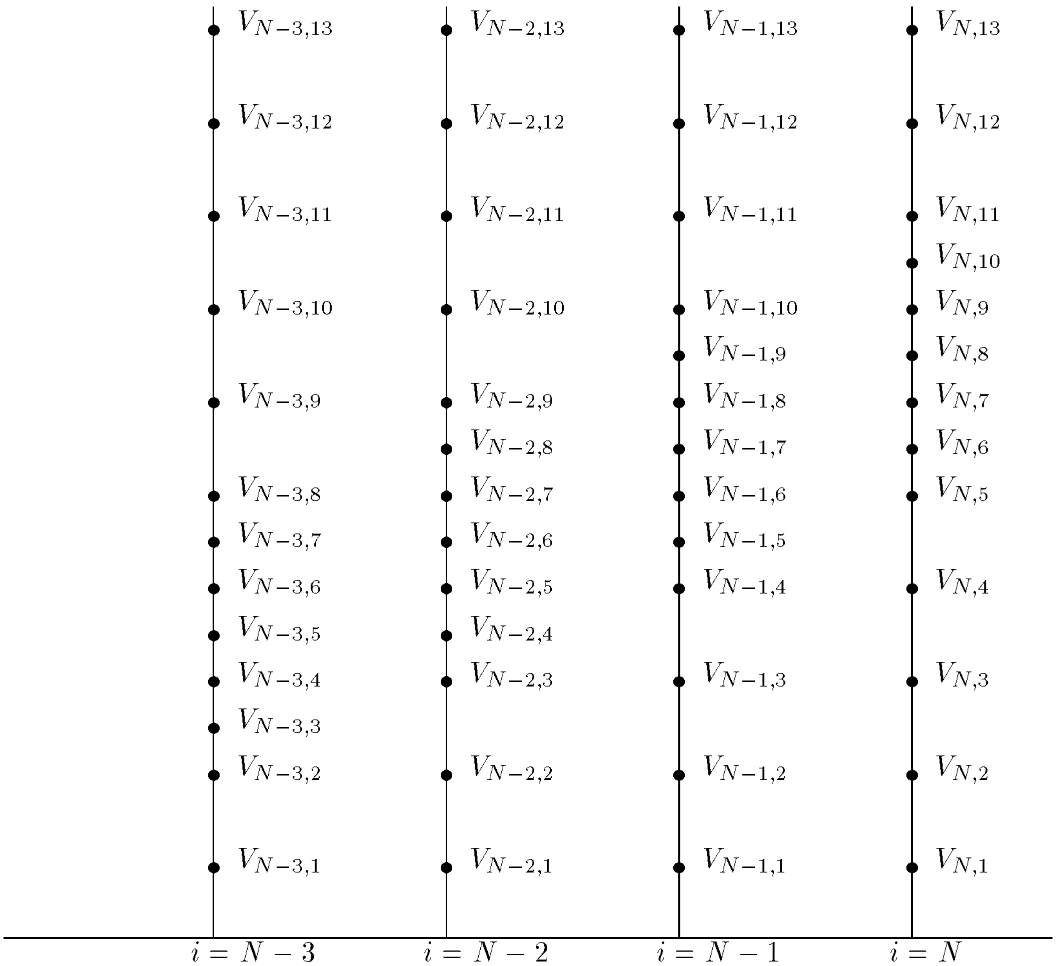

Figure 2 : Remeshing at different i

Table 1: Experiments on K and P.

$T = 1, c = 0.05$ $\sigma^2 = 0.03, r = 0.1, \beta = 0.08$	$P = 0.5$ $K = 0.1$	$P = 0.5$ $K = 0.2$	$P = 1$ $K = 0.1$	$P = 1$ $K = 0.2$
V^*				
Explicit	0.46	0.55	0.83	0.92
Log Explicit	0.47	0.55	0.82	0.90
Implicit	0.45	0.54	0.82	0.91
Dynamic Implicit	0.45	0.54	0.82	0.91
$B(V = 1)$				
Explicit	0.476	0.475	0.849	0.781
Log Explicit	0.476	0.475	0.846	0.776
Implicit	0.476	0.475	0.846	0.779
Dynamic Implicit	0.476	0.475	0.849	0.780
Time(sec)				
Explicit	3145	3086	2924	2967
Log Explicit	110	108	102	103
Implicit	230	227	198	234
Dynamic Implicit	48	47	47	47

Table 2: Experiments on σ and T.

	$\sigma^2 = 0.03$			$\sigma^2 = 0.1$		
	V^*	Time	work	V^*	Time	work
$T = 1$						
Log Explicit	0.90	100	2.08	0.81	336	6.86
Implicit	0.91	193	4.02	0.82	230	4.69
Dynamic Implicit	0.92	48	1	0.82	49	1
$T = 2$						
Log Explicit	0.81	202	2.19	0.71	678	7.2
Implicit	0.80	362	3.93	0.70	401	4.27
Dynamic Implicit	0.82	92	1	0.70	94	1
$T = 5$						
Log Explicit	0.69	513	2.26	0.57	1738	7.56
Implicit	0.67	938	4.13	0.54	985	4.28
Dynamic Implicit	0.68	227	1	0.54	230	1
$T = 10$						
Log Explicit	0.62	1028	2.32	0.51	3584	7.84
Implicit	0.60	1759	3.97	0.47	1998	4.37
Dynamic Implicit	0.60	443	1	0.47	457	1
$T = 30$						
Log Explicit	0.56	3030	2.25	0.47	10558	7.94
Implicit	0.54	5420	4.02	0.42	6045	4.55
Dynamic Implicit	0.54	1346	1	0.42	1330	1